Radioactive Dice - Background

Introduction

STRUCTURE

The following pages contain a summary of the core concepts at play in this exercise. Some or all of these concepts may be relevant to your classroom objectives. The pages of this guide have a three column **structure**, whereby each concept gets a keyword, an explanation, and illustration (or example).

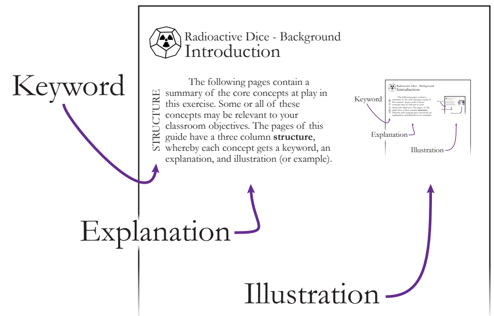


Table of Contents

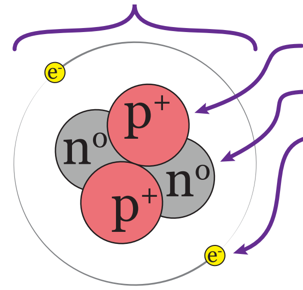
Radioactive Dice - Background

Anatomy of an Atom

ATOM

Atoms have a nucleus - a core - made of some number of protons and neutrons, and are surrounded by a cloud of electrons.

This is an atom. It has



2 protons,
2 neutrons, and
2 electrons.

CHARGE

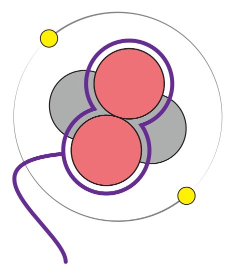
Electrons have a negative **charge**, protons have an equally strong positive charge, and neutrons have no charge. If an atom has an unequal number of electrons and protons, the atom itself has a net charge. Charged atoms and molecules are called ions. Oppositely charged ions might come together like magnets to form ionic bonds, or they might flow from one place to another in an electromagnetic field.

This atom's total **charge** is zero, so it is not an ion.

$$\begin{array}{r} 2\times(+1) \\ 2\times(0) \\ +\ 2\times(-1) \\ \hline \rightarrow \mathbf{0} \end{array}$$

ELEMENT

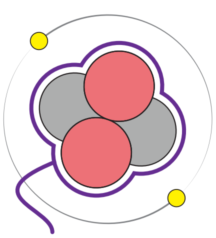
The number of protons an atom contains determines essentially everything about how that atom behaves chemically. For this reason, we classify atoms into **elements** based on this number.



This atom has 2 protons, so it is classified as the **element** helium (He).

ISOTOPE

Two atoms belonging to the same element might have a different number of neutrons, resulting in slightly different versions of the same element. We call these versions "**isotopes.**" Different isotopes of the same element behave mostly the same way chemically, but are a little heavier or lighter.

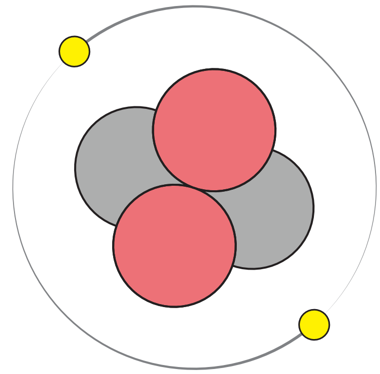


This atom has 4 total protons and neutrons, so it is classified as the **isotope** of helium called helium-4, abbreviated ^{4}He.

Radioactive Decay

STABILITY

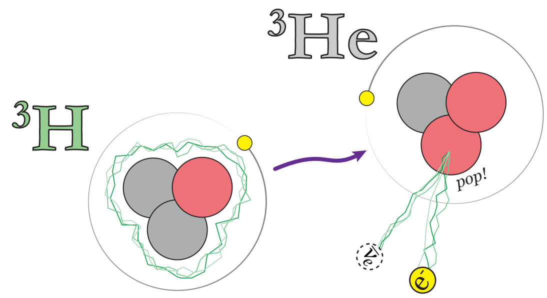
The number and configuration of protons and neutrons in the nucleus of an atom also determines if that atom is **stable**. An unstable atom is said to be radioactive, and might spontaneously undergo radioactive decay, emitting radiation and transforming into a different isotope or even a different element.



Helium-4 is considered stable because it has never been observed to undergo radioactive decay.

HALF-LIFE

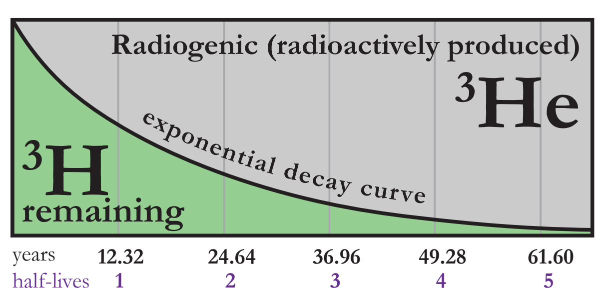
Radioactive decay is a random process, but it is more or less likely in different isotopes. Radioactive isotopes can be very unstable, lasting for only fractions of a second, or they can be only a little unstable, lasting for billions of years. We measure this instability in terms of **half-life**. The half-life of a radioactive isotope is an interval of time within which an atom has a 50-50 chance of decaying. Very unstable isotopes have short half-lives. Less unstable ones have long half-lives.



Unlike helium-4, the isotope hydrogen-3 *is* radioactive. ^{3}H decays into ^{3}He with a half-life of 12.32 years. That means that if you watch an atom of ^{3}H for 12.32 years, there's a 50-50 chance you'll see it spontaneously transform into ^{3}He. This happens when one of its neutrons turns into a proton by kicking out an electron and an electron anti-neutrino.

EXPONENTIAL DECAY

Of course, we almost never consider one atom at a time, but the *septillions* of atoms that make up "stuff." If you watch a large number of radioactive atoms over time, they will decay according to the **exponential decay** function: a function of time and half-life. 50% of the remaining atoms will decay every half-life.



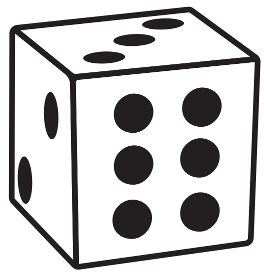
Radioactive Dice - Background

Probability Theory

PROBABILITY

The **probability** of an event is the chance that it will occur. Probability theory is the mathematical description of probability. We encounter probability every day, and have good intuition for things like coin tosses, but probability *theory* lets us determine the chance of more complicated events like a coin landing heads-up three times in a row.

Consider the rolling of a standard die.



There are six possible, equally likely outcomes.

The die could land on 1, 2, 3, 4, 5, or 6.

P(e)

In probability theory, the probability of an event "e" occurring is symbolized **P(e)**. When e is impossible, we say P(e)=0. When e is certain, we say P(e)=1. When an event is neither certain, nor impossible, the value of its probability is between 0 and 1. Probabilities are often expressed as decimals, fractions, or percentages.

Events are denoted by variables.

t: "the die lands on 3"

s: "the die lands on 6"

"P(e)" reads "the probability of e."

P(t)= 1/6 = 16.6% = .16

P(s)= 1/6 = 16.6% = .16

"OR"

A pair of events can be mutually exclusive, meaning they can't happen at the same time. In these cases, the combined probability of one event **or** the other happening is the sum of their individual probabilities.

t and **s** are mutually exclusive. There is no chance they will both happen.

So, **P(t or s)** = 1/6 **+** 1/6 = 1/3

Radioactive Dice - Background

Probability Theory

“NOT”

The complement of an event comprises the set of all possible outcomes *other than* that event. The probability of one of these outcomes occurring is 1-P(e). The complement of e could be thought of as “**not**-e,” and is often symbolized “$\bar{e}$”.

The complement of **t** is symbolized **$\bar{t}$**, which you could read as “not-t”. It stands for the event “the die lands on 1, 2, 4, 5, or 6”

$$\mathbf{P(not\ t) = P(\bar{t}) = 1\text{-}P(t)} = 5/6$$

“AND”

A pair of events can be independent, meaning the occurrence of one does not affect the probability of the other. For example, the results of series of dice rolls are independent, because past rolls don’t affect future rolls. In these cases, the overall probability of *both* events (i.e. the first event **and** the second event) occurring is the *product* of their probabilities.

Imagine a series of n dice rolls.
t_n: “The die lands on 3 on the n[th] roll.”

t_1 and t_2 are independent, so the chance of rolling two 3’s in a row is:

$$\mathbf{P(t_1\ and\ t_2)} = 1/6 \times 1/6 = 1/36$$

TRANSLATION

Knowing only how to say “**or**,” “**not**,” and “**and**,” we can calculate chances that appear to be beyond our abilities to describe if we can be clever translators. For example, we can’t symbolize...

“two coins land on the same side,”

but we can symbolize...

“[coin 1 lands on heads **and** coin 2 lands on heads] **or** [coin 1 does not land on heads **and** coin 2 does not land on heads]”

Which might be symbolized:

$$[P(h_1)\times P(h_2)]+[(P(\bar{h}_1)\times P(\bar{h}_2)] =$$
$$.5 \times .5 + .5 \times .5 = \mathbf{.5}$$

a: “The die lands on 3 at least once in a series of three rolls.”

What is the value of **P(a)**? We don’t know how to directly symbolize this, but we can do it if we manage to say “at least once” in terms of “not,” “and,” and “or.” One way to do this is...

$$\mathbf{P(a) = 1 - P(\bar{t}_1\ and\ \bar{t}_2\ and\ \bar{t}_3)}$$

which might read “The probability of **a** is the probability of **not** rolling **not-three** on the **first and second and third** roll.”

$$1 - (5/6 \times 5/6 \times 5/6) = \mathbf{42\%}$$

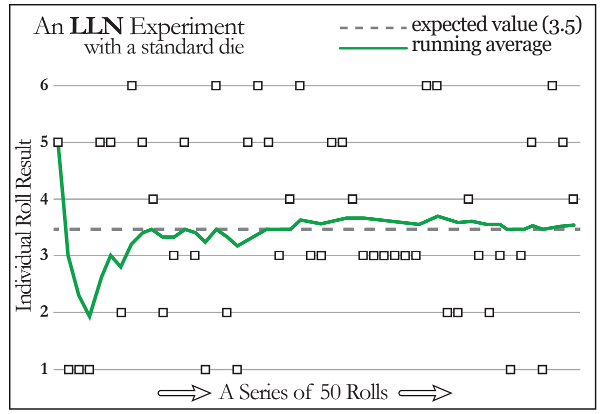
Radioactive Dice - Background

Dice & Exponential Decay

LAW OF LARGE NUMBERS (LLN)

The Law of Large Numbers

(LLN) is a principle of statistics that states that while small numbers of probabilistic entities like dice or radioactive atoms behave unpredictably, larger groups behave more predictably - tending to converge as a group on an expected behavior.

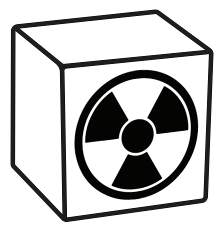


DECAY OF A SINGLE DIE

A single "radioactive die" like the one pictured to the right has a 1/6 chance of "decaying" on a given roll. In order to survive two rolls, it must survive one roll and then another roll - an event with a probability of $5/6 \times 5/6 = 25/36$.

The chance that such a die will last "**r**" rolls is $5/6 \times 5/6 \times$.... 5/6 repeated **r** times, or $(5/6)^r$.

Consider the rolling of a this special die:



It has one face marked ☢, and five blank faces.

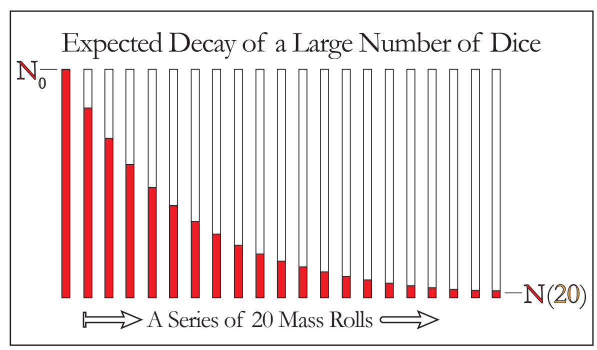
We will say that it "decays" when it lands on ☢.

$N(r) = N_0 \cdot (5/6)^r$

A single such die behaves unpredictably, but because of the LLN, about 5/6 of a large population of radioactive dice *will* survive a given roll. A mathematical description of this decay might look like:

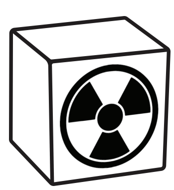
$N(r) = N_0 \cdot (5/6)^r$, read:

"The population (after so many rolls) is the initial population multiplied by (5/6) once for every roll."



FIVE-SIXTHS-LIFE

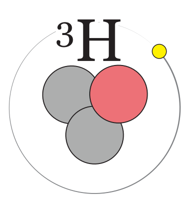
$N(r) = N_0 \cdot (5/6)^r$ is an exponential decay function. Functions of this same form govern the decay of radioactive atoms. However, those functions tend to be written in terms of half-life, which allows us to compare, at a glance, the relative stabilities of different radioisotopes. Our equation, written in terms of "five-sixths-lives," is not as useful. We'll need logarithms to convert it.



"five-sixths-life" in terms of rolls

rolls elapsed

$N(r) = N_0 \cdot (5/6)^{r\,/\,1\text{ roll}}$



$N(t) = N_0 \cdot (1/2)^{t\,/12.32\text{ yr}}$

years elapsed

half-life in terms of years

Radioactive Dice - Background

Dice & Exponential Decay

CONVERTING TO HALF-LIFE

We want to know the half-life (in terms of rolls) of a population of radioactive dice. We can see on the red graph that it takes between 3 and 4 rolls for N(r) to reach 50% of N_0, but to solve for the exact value, we need to solve $N(r)/N_o = (5/6)^r = 50\%$ for r. Basically, we are asking:
"What power r would turn 5/6 into 1/2?"

By solving for r, we find that $N(r)/N_o = 50\%$ when r = 3.8. In other words, the half-life of a six-sided die is 3.8 rolls. Now, by coming up with a fictitious **conversion factor** between rolls and years, you can produce populations of radioactive dice that behave just like any given radioisotope.

Math	English
$(5/6)^r = .5$	"5/6 to the power r is .5."
$\text{Log}_{5/6}.5 = r$	"The power that turns 5/6 into .5 is r."

$\text{Log}_{5/6}.5 = r = \mathbf{3.8}$ ← half-life in terms of rolls!

$$\frac{\mathbf{12.32}\text{ yr.}}{\mathbf{3.8}\text{ rolls}} = \frac{\mathbf{3.24}\text{ yr.}}{\mathbf{1}\text{ roll}}$$

(12.32 yr.: half-life of ^3H; 3.8 rolls: half-life of a six-sided die)

So, when 1 roll represents 3.24 years elapsing, a six-sided die behaves just like an atom of ^3H!

RADIOMETRIC DATING

Now that we are capable of setting up an environment in which a group of dice faithfully represents a group of radioactive atoms, we can do more than just watch those dice decay. We can examine a partially decayed group and determine its "age" based on how many dice have decayed and how many dice remain. This rearranging of the exponential decay function is the basis for **radiometric dating**.

Consider 36 dice that have decayed into ^3He and **14 ^3H dice that remain.**

$N(r)/N_o = (1/2)^{r\,/\,3.8\text{ rolls}}$

$14/(36+14) = (1/2)^{r\,/\,3.8\text{ rolls}}$

$.28 = (1/2)^{r\,/\,3.8\text{ rolls}}$

$\text{Log}_{(1/2)}.28 = \mathbf{r} / 3.8$

$\mathbf{r} \approx 7$ rolls old, or

$7 \times 3.24 = 22.68$ years old